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Non-classical states of light and canonical transformations

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Abstract. Representations of nonlinear non-bijective canonical transformations in quantum mechanics are discussed. Due to the non-bijectivity the classical phase space has a Riemann sheet structure, and a family of partial isometries translating this structure into quantum mechanics is constructed. If a unitary representation is required, a new variable—the ambiguity spin—has to be introduced in order to recover bijectivity following the approach of Moshinsky and co-workers. This new degree of freedom is analysed in terms of multiboson operators. The application of this formalism to some non-classical states of light is discussed.

1. Introduction

It is well known that canonical transformations are a powerful and elegant technique for solving problems in classical mechanics [1]. This subject was also important for the development of the foundations of quantum mechanics [2], but this interest lessened after the objective was achieved.

It seems to be generally accepted that canonical transformations are represented in quantum mechanics by unitary operators. For the case of linear canonical transformations, a celebrated theorem of Von Neumann [3] is usually invoked. This group of linear canonical transformations has been extensively discussed in the literature [4] and has a host of applications in many branches of physics, going from problems in nuclear clustering theory [5], groups of accidental degeneracies etc [6], to the theory of superconductivity through the famous Bogoliubov transformation [7]. In quantum optics, perhaps one of the most relevant is the characterization of squeezed states as generalized coherent states for this group [8]. We wish to stress that the Von Neumann theorem holds only when the system has finite degrees of freedom. In the case of an infinite number of degrees of freedom such an operator may not exist, since physically equivalent observables, i.e. realizing the same algebra of commutation relations, are not necessarily unitarily equivalent [9].

For more complicated situations one is usually referred to Dirac's classic book [10]. However, the programme outlined by Dirac works only under the restriction that the transformation relate operators with the same spectrum. When it is not the case, as occurs frequently, the procedure is meaningless.

Moshinsky and co-workers [11] have discussed extensively what happens when one considers canonical transformations in which this restriction does not hold. In particular they have fully analysed nonlinear and non-bijective canonical transformations. In this paper we are interested in the specific example of a transformation that relates

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a harmonic oscillator of unit frequency with another of integer frequency κ. This example may seem trivial, but it contains all the basic difficulties of the problem. The transformation becomes non-bijective, a fact intimately related with the difference between the spectra of both Hamiltonians, and no such unitary operator can exist.

If we require a unitary representation, it seems necessary to introduce a new variable—first called by Plebański *ambiguity spin*, and whose applications to the representations of non-bijective canonical transformations have been worked out by Moshinsky and co-workers [12]—that restore bijectivity and at the same time equalize the spectra.

Up to now no clear physical interpretation of this ambiguity spin has been made. In this work we attempt to reinterpret this variable in connection with the multiphoton operators [13] recently introduced to produce non-Gaussian squeezing. Among the quantum properties of the squeezed states perhaps the most important are the reduction of the uncertainty in one quadrature component of the field below the vacuum, and the presence in some cases of sub-Poissonian statistics.

The schemes to generate squeezed states are essentially based on nonlinear optical effects where the fields interact with matter characterized, in general, by a kth-order susceptibility, which corresponds to produce k-photon states, and multiphoton processes are involved in the interaction Hamiltonian.

The generalization of k-photon squeezing for $k > 2$ runs into difficulties that can be overcome with the introduction of these multiphoton operators. A clear understanding of the problem can provide an interesting tool for handling many problems occurring in connection with some non-classical properties of special photon states.

2. Representations of non-bijective canonical transformations

In this section we intend to summarize the main concepts introduced by Moshinsky and co-workers [12] in connection with the representations in quantum mechanics of non-bijective canonical transformations. We try to introduce these concepts in such a way that they show their connection with the multiboson formalism we shall deal with in section 3.

Focusing on the case of the transformation relating an oscillator of unit frequency with another of frequency κ, with κ an integer, its implicit definition can be expressed as [14]

$$\begin{aligned} \tfrac{1}{2}(p^2+q^2) &= \tfrac{1}{2}\kappa(\overline{p}^2+\overline{q}^2) \\ \kappa \tan^{-1}\frac{p}{q} &= \tan^{-1}\frac{\overline{p}}{\overline{q}} \end{aligned} \qquad (2.1)$$

where (q,p) and $(\overline{q},\overline{p})$ stand for the old and new coordinates in phase space. This transformation is essentially a dilatation in the action-angle phase space. A sector of angle $2\pi/\kappa$ in the (q,p) plane is mapped on the full plane $(\overline{q},\overline{p})$, thus it is non-bijective and to retrieve bijectivity the phase plane $(\overline{q},\overline{p})$ must have κ sheets connected along the cut in the positive real axis.

Another approach that is easily translated into quantum mechanics begins by considering the set of points in the original phase space mapped on the same point of the new plane. These points (q_η, p_η) are

$$\begin{aligned} q_\eta &= q\cos(2\pi\eta/\kappa) - p\sin(2\pi\eta/\kappa) \\ p_\eta &= q\sin(2\pi\eta/\kappa) + p\cos(2\pi\eta/\kappa) \end{aligned} \qquad (2.2)$$

where $\eta = 0, 1, \ldots, \kappa - 1$.

As we can see these points are connected by a group of linear canonical transformations isomorphic to the cyclic group C_κ. This group, intimately related with the non-bijectivity, is called the ambiguity group. We can retain the idea of a single sheet, but characterizing the functions in the (q,p) plane by components invariant under the ambiguity group. Since each component maps unambiguously, functions on the $(\overline{q}, \overline{p})$ plane must be vector functions. The index labelling these components is called ambiguity spin.

In quantum mechanics a representation of a canonical transformation is a *unitary operator* U that translates the classical relations between observables in the original and new phase spaces into relations between *operators* in the corresponding Hilbert spaces $\mathcal{H}$ and $\overline{\mathcal{H}}$. For the transformation (2.1), the relation between operators we want to establish is

$$\begin{aligned} U\ a^\dagger a\ U^\dagger &= \kappa \overline{a}^\dagger \overline{a} \\ U\ E^\kappa\ U^\dagger &= \overline{E}. \end{aligned} \tag{2.3}$$

The first equality is the relation between the two Hamiltonians described as usual in terms of the creation and annihilation operators. The second one is the closest translation into quantum mechanics of the classical phase relation in terms of the phase operator of Susskind and Glogower [15], defined as $E = 1/\sqrt{n+1}\ a$.

If the spectrum of both Hamiltonians were the same, we could easily construct the unitary operator of the transformation following Dirac's prescription [10]. However in our case $a^\dagger a$ and $\kappa \bar{a}^\dagger \bar{a}$ do not have the same spectrum and no such unitary operator U exists.

Instead of a unitary operator we can try to give isometric mappings restricted to certain subspaces of $\mathcal{H}$ using the concept of ambiguity group.

In the quantum case, the linear canonical transformations $V(\eta)$ in $\mathcal{H}$ that leave invariant $a^\dagger a$ and E^κ in (2.3) are

$$V(\eta) = \exp \mathrm{i}\left(\frac{2\pi}{\kappa}\eta a^\dagger a\right) \qquad \eta = 0, \ldots, \kappa - 1 \tag{2.4}$$

that is, a representation of the cyclic group C_κ defined as well in the classical case by (2.2).

Thus, in order to find subspaces that could be related isometrically with $\overline{\mathcal{H}}$ satisfying (2.3) we must restrict to the subspaces of $\mathcal{H}$ where the action of the ambiguity group becomes a constant phase factor, i.e. subspaces carrying the unitary representations of the group. These subspaces $\mathcal{H}_\lambda$ are spanned by $|\kappa n + \lambda\rangle$, $n = 0, \ldots, \infty$, and we have that $\mathcal{H}$ splits as

$$\mathcal{H} = \bigoplus_{\lambda=0}^{\kappa-1} \mathcal{H}_\lambda. \tag{2.5}$$

Now the family of partial isometries U_λ mapping isometrically the subspaces $\mathcal{H}_\lambda$ onto $\overline{\mathcal{H}}$ verifying (2.3) are easily constructed following Dirac's programme, and we get

$$U_\lambda = \sum_n |n)\ \langle \kappa n + \lambda| \qquad \lambda = 0, \ldots, \kappa - 1 \tag{2.6}$$

where $|n)$ and $|n\rangle$ are the number states in both Hilbert spaces.

In the coordinate representation the matrix elements of U_λ take the form

$$\langle\bar{q}|\, U_\lambda |q\rangle = \sum_{n=0}^{\infty} \psi_n{}^*(\bar{q})\ \psi_{\kappa n+\lambda}(q) \tag{2.7}$$

where $\psi(q)$ are the normalized solutions of the Schrödinger equation for an oscillator of unit mass and frequency. This form is the same obtained by Kramer *et al* [12] with a different approach.

This family of partial isometries seems the closest relation between our canonical transformation and unitary operators in quantum mechanics.

If now we require the transformation to be unitary it has been proposed to enlarge the final Hilbert space to recover bijectivity. The simplest way to do this is to consider as the final space not $\overline{\mathcal{H}}$ but $\overline{\mathcal{H}} \otimes \mathcal{V}$, where $\mathcal{V}$ is some finite-dimensional space that translates into quantum mechanics the multicomponent structure of the functions in phase space, and that might be called ambiguity spin space. As pointed out by Plebański [16] this corresponds to extending the family of semi-unitary operators U_λ to an isometry. The role played by this space $\mathcal{V}$ is to allow each $\mathcal{H}_\lambda$ to have a different image and provide a variable that equals both spectra. Taking now $\overline{\mathcal{H}} \otimes \mathcal{V}$ as the final space we can construct a truly unitary operator $U : \mathcal{H} \longrightarrow \overline{\mathcal{H}} \otimes \mathcal{V}$

$$U = \sum_{n,\lambda} |e_\lambda\rangle |n)\langle \kappa n + \lambda| \tag{2.8}$$

where $|e_\lambda\rangle$ is a basis in $\mathcal{V}$.

Now the final transformation is

$$\begin{aligned} U\ a^\dagger a\ U^\dagger &= \kappa \bar{a}^\dagger \bar{a} + \sum_\lambda \lambda |e_\lambda\rangle\langle e_\lambda| \\ U\ E^\kappa\ U^\dagger &= \overline{E}. \end{aligned} \tag{2.9}$$

To recover the correspondence between canonical transformations and unitary operators, it has been necessary to add a new degree of freedom that, until now, has not received a clear physical interpretation. Note that the final space and the original one are clearly discriminated, even if in principle $\mathcal{H} = \overline{\mathcal{H}}$.

In the next section we show that with the formalism of multiboson operators we can identify $\overline{\mathcal{H}} \otimes \mathcal{V}$ with our physical space $\mathcal{H}$ and, in this way, give an interpretation of the ambiguity spin.

3. Multiphoton operators

In this section we try to relate the non-bijective canonical transformation problem with the formalism of the multiboson operators. We shall use the same notation used before for what may be considered as a different question. We think that the final result justifies this choice.

As is well known, the abstract commutation relations

$$[a, a^\dagger] = I \tag{3.1}$$

have the usual irreducible representation in the familiar Fock space $\mathcal{H}$ with the action of the operators defined as

$$a|n\rangle = \sqrt{n}|n-1\rangle \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{3.2}$$

with $a|0\rangle = 0$, $|n\rangle$ being a complete orthonormal basis of $\mathcal{H}$. Any other irreducible representation of (3.1) by closed densely defined operators in a Hilbert space is unitarily equivalent to it [17].

In this Fock space of the single Bose operator a, we can define the generalized Bose operators or multiboson operators $a_{(\kappa)}$ as

$$\begin{aligned} a_{(\kappa)}|\kappa n + \lambda\rangle &= \sqrt{n}|\kappa(n-1)+\lambda\rangle \\ a^\dagger_{(\kappa)}|\kappa n + \lambda\rangle &= \sqrt{n+1}|\kappa(n+1)+\lambda\rangle \end{aligned} \tag{3.3}$$

where $\lambda = 0, \ldots, \kappa - 1$, with commutation relation

$$[a_{(\kappa)}, a^\dagger_{(\kappa)}] = I. \tag{3.4}$$

These equations lead us to interpret $a_{(\kappa)}$ as an annihilation operator of κ bosons simultaneously. However, it should be noted that while $a_{(1)} = a$, $a_{(\kappa)} \neq a^\kappa$ for $\kappa \geq 2$.

Perhaps the clearest expression of $a_{(\kappa)}$ in terms of a and $a^\dagger$ can be obtained from the relation

$$E^\kappa = E_{(\kappa)} \tag{3.5}$$

where $E_{(\kappa)}$ is defined as

$$E_{(\kappa)} = \frac{1}{\sqrt{a^\dagger_{(\kappa)}a_{(\kappa)} + 1}}\, a_{(\kappa)}. \tag{3.6}$$

Associated with these operators we have the corresponding canonical variables $(q_{(\kappa)}, p_{(\kappa)})$ defined as usual in terms of $a_{(\kappa)}$ and $a^\dagger_{(\kappa)}$.

With these definitions $\mathcal{H}$ splits naturally into a direct sum of subspaces $\mathcal{H}_\lambda$ invariant under the action of the multiboson operators. These $\mathcal{H}_\lambda$ are defined in the same way as in the previous section, and now $(a_{(\kappa)}, \mathcal{H}_\lambda)$ are irreducible representations of (3.1).

This decomposition of $\mathcal{H}$ shows that the canonical transformation problem and these operators must be related. In order to get the connection between the two formalisms we can express the previous direct sum as a tensor product of Hilbert spaces, one isomorphic to any of the summands and another with dimension the length of the sum. Calling $\overline{\mathcal{H}}$ and $\mathcal{V}$ to these spaces we have the following isomorphism $\mathcal{H} \approx \overline{\mathcal{H}} \otimes \mathcal{V}$. If we take $|n\rangle$ as a complete orthonormal basis in $\overline{\mathcal{H}}$ and $|e_\lambda\rangle$ as a basis in $\mathcal{V}$, this isomorphism can be realized through the unitary operator U

$$U = \sum_{n,\lambda} |e_\lambda\rangle|n\rangle\langle\kappa n + \lambda|. \tag{3.7}$$

With $\bar{a}$ defined by

$$\bar{a} = U\ a_{(\kappa)}\ U^\dagger \tag{3.8}$$

the pair $(\bar{a}, \overline{\mathcal{H}})$ is an irreducible representation of (3.1).

It is clear that this unitary operator and (2.8) are the same transformation and this is the reason we use the same notation $\overline{\mathcal{H}}$ here for the quotient space and for the new Hilbert space in section 2.

We can say after this identification that it is not necessary to enlarge the final Hilbert space introducing spurious variables in order to have the unitary transformation (2.8), from the canonical transformation (2.1). The invariance of the Poisson brackets does not guarantee that the new variables $\bar{q}$ and $\bar{p}$ are a complete set of coordinates in phase space.

In quantum mechanics this fact is reflected, as the previous isomorphism shows, in that we cannot obtain from the $(\bar{q}, \bar{p})$ operators, or in the same way from $(q_{(\kappa)}, p_{(\kappa)})$, a complete set of commuting observables. In fact we can find operators that commute with both pairs and are not constants.

We can give another approach to this identification. If we express the Hamiltonian of unit frequency in terms of multiboson operators we have

$$a^{\dagger} a = \kappa a^{\dagger}_{(\kappa)} a_{(\kappa)} + \sum_{\lambda} \lambda P_{\lambda} \tag{3.9}$$

where P_{λ} is the projector onto the subspace $\mathcal{H}_{\lambda}$. If we translate (3.9) to the $|n\rangle|e_{\lambda}\rangle$ basis, we have just the first expression in (2.9). We can see that $\bar{a}$ evolves with frequency κ, but can not be unitarily equivalent to a, because it is not an irreducible representation of (3.1) in $\mathcal{H}$.

4. Non-classical states of light

The example presented above is very simple. However this kind of identification of variables in phase space or the corresponding subspaces in the Hilbert space, can be useful not only in connection with canonical transformations, but with other problems.

In quantum optics a proper definition of variables can be used for the definition of new classes of non-classical states of light or for the generalization of the ones previously introduced.

As an example we can consider the multiphoton squeezed states [18]. They were introduced to overcome the difficulties that arise in the naive generalization of one- and two-photon coherent states to higher photon orders. It is well known that coherent and squeezed states are generated through the action on the vacuum of unitary operators in the form of an exponential of polynomials quadratic in a and $a^{\dagger}$. It has been pointed out by Fisher *et al* [19] the impossibility of the generalization of this action to powers higher than two. However the simple one-photon coherent states have been generalized to κ-photon states by means of the multiphoton operators defined in the previous section.

The multiphoton squeezed states are defined as

$$|\alpha_{(\kappa)}\rangle \equiv \exp(\alpha a^{\dagger}_{(\kappa)} - \alpha^{*} a_{(\kappa)})|0\rangle. \tag{4.1}$$

Their expression in the number basis is simply given by

$$|\alpha_{(\kappa)}\rangle = \exp -\left(|\alpha|^2/2\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \, |\kappa n\rangle. \tag{4.2}$$

Perhaps the most interesting property of these states is that their probability distribution functions are not Gaussian. They are not minimum uncertainty states (MUS), but present squeezing for $\kappa = 2$. It is clear that these kind of states arise in processes involving κ-photon interactions.

We can express (4.1) not in $\mathcal{H}$ but in $\overline{\mathcal{H}} \otimes \mathcal{V}$, by means of the operator U

$$U|\alpha_{(\kappa)}\rangle = U \exp(\alpha a^{\dagger}_{(\kappa)} - \alpha^* a_{(\kappa)})U^{\dagger}\ U|0\rangle = \left[\exp(\alpha\bar{a}^{\dagger} - \alpha^*\bar{a})|0)\right]|e_0\rangle. \tag{4.3}$$

This expression in the $|n)|e_\lambda\rangle$ basis shows that these states are nothing but coherent states for the $\overline{q}$ and $\overline{p}$ variables. In the same way we could identify in these variables other states defined in terms of the multiphoton operators.

Up to now the space $\mathcal{V}$ has only been used to index subspaces of $\mathcal{H}$. However it can be used to extend to the whole $\mathcal{H}$ the definition of states and operators made in finite-dimensional subspaces as occurs with the phase operator of Pegg and Barnett and the realizations of SU(2) in terms of bosonic operators, so widely used in quantum optics.

Let us focus on the Holstein–Primakoff [20] realizations of the SU(2) algebra. The abstract commutation relations of SU(2) can be realized in terms of the bosonic operators $a, a^{\dagger}$, but *restricted* to a finite-dimensional subspace of $\mathcal{H}$. If we take the $(2\sigma+1)$-dimensional subspace spanned by the number states $|n\rangle$, where n now ranges from 0 to 2σ, the infinitesimal generators take the form:

$$\begin{aligned} J_- &= \sqrt{2\sigma - a^{\dagger}a}\ a \\ J_+ &= a^{\dagger}\ \sqrt{2\sigma - a^{\dagger}a} \\ J_z &= a^{\dagger}a - \sigma. \end{aligned} \tag{4.4}$$

It is easy to see that these definitions depend on the finite-dimensional subspace chosen. We can use the previous isomorphism $\mathcal{H} \approx \overline{\mathcal{H}} \otimes \mathcal{V}$ to give another definition valid in the whole space.

Any $(2\sigma+1)$-dimensional subspace of $\mathcal{H}$ is isomorphic to a subspace of $\overline{\mathcal{H}} \otimes \mathcal{V}$ of the form $|\psi) \otimes \mathcal{V}$ if the dimension of $\mathcal{V}$ is $2\sigma+1$. In particular, we can take $|\psi)$ to be any number state in $\overline{\mathcal{H}}$.

If now $\mathcal{V}$ carries a κ-dimensional representation of SU(2) (calling the infinitesimal generators $\boldsymbol{j}$, the Casimir $\boldsymbol{j}^2$ takes the value $\frac{1}{4}(\kappa^2-1)$) we can chose the basis $|e_\lambda\rangle$ to be the eigenvectors of j_z, that is, $j_z|e_\lambda\rangle = [\lambda - (\kappa-1)/2]\,|e_\lambda\rangle$.

Translating these operators $\boldsymbol{j}$ from $\overline{\mathcal{H}} \otimes \mathcal{V}$ to $\mathcal{H}$, we have for example for j_-

$$U^{\dagger}(I \otimes j_-)U = \sum_{n,\lambda} \sqrt{\lambda(\kappa-\lambda)}|\kappa n + \lambda - 1\rangle\langle\kappa n + \lambda|. \tag{4.5}$$

This definition splits $\mathcal{H}$ in a sum of κ-dimensional representations of SU(2), each one labelled by the multiboson number n. We recover the usual definition (4.4) taking $n = 0$ and $\kappa = 2\sigma + 1$.

This can be viewed as an example of the discussion after (3.7). We can add to $\overline{q}$ or $\overline{p}$ (or, in much the same way to $q_{(\kappa)}$ or $p_{(\kappa)}$) J_z (defined as $J_z = U^{\dagger}\ (I \otimes j_z)\ U$) to form a complete set of commuting observables.

We can see the above definition in the SU(2) case as complementary with the case of the multiphoton states. In the later $\mathcal{V}$ remains fixed while in the former this happens with $\overline{\mathcal{H}}$, that is not involved in the definitions.

With the definitions made above, if we take $\kappa = 2$ the operators a and $a^\dagger$ are mapped by U to the operators acting in $\overline{\mathcal{H}} \otimes \mathcal{V}$

$$\begin{aligned} UaU^\dagger &= \sqrt{2\overline{a}^\dagger\overline{a}+1}\; j_- + \sqrt{2}\;\overline{a}\; j_+ \\ Ua^\dagger U^\dagger &= \sqrt{2\overline{a}^\dagger\overline{a}+1}\; j_+ + \sqrt{2}\;\overline{a}^\dagger\; j_- . \end{aligned} \tag{4.6}$$

In the same way the infinitesimal generators $\frac{1}{2}(a^\dagger)^2$, $\frac{1}{2}a^2$ and $\frac{1}{4}(a^\dagger a + aa^\dagger)$ of SU(1,1) can be expressed as

$$\begin{aligned} \tfrac{1}{2}Ua^2U^\dagger &= \begin{pmatrix} \overline{a}\sqrt{\overline{a}^\dagger\overline{a}+\frac{1}{2}} & 0 \\ 0 & \sqrt{\overline{a}^\dagger\overline{a}+\frac{1}{2}}\;\overline{a} \end{pmatrix} \\ \tfrac{1}{2}U(a^\dagger)^2U^\dagger &= \begin{pmatrix} \sqrt{\overline{a}^\dagger\overline{a}+\frac{1}{2}}\;\overline{a}^\dagger & 0 \\ 0 & \overline{a}^\dagger\sqrt{\overline{a}^\dagger\overline{a}+\frac{1}{2}} \end{pmatrix} \\ \tfrac{1}{4}U(a^\dagger a + aa^\dagger)U^\dagger &= \begin{pmatrix} \overline{a}^\dagger\overline{a}+\frac{3}{4} & 0 \\ 0 & \overline{a}^\dagger\overline{a}+\frac{1}{4} \end{pmatrix}. \end{aligned} \tag{4.7}$$

As we can see the commutation relations do not involve the variable λ and we can consider the operators in the diagonal in the right-hand side of the above relations acting not in $\overline{\mathcal{H}}$ but in $\mathcal{H}$. In this way we obtain two realizations of the algebra different from the first one.

The formalism of Pegg and Barnett [21], describing the phase properties of a single-mode field, deals with a finite-dimensional subspace of $\mathcal{H}$. If we take its dimension to be κ, this space is spanned by the number states $|n\rangle$ with $n = 0, \ldots, \kappa - 1$. The phase states are defined as

$$|\theta\rangle = \frac{1}{\sqrt{\kappa}} \sum_{n=0}^{\kappa-1} \exp \mathrm{i}(n\theta)|n\rangle. \tag{4.8}$$

Their properties becomes the expected properties for a well behaved phase state in the limit $\kappa \longrightarrow \infty$ (in the same way that occurs with the phase states of Susskind and Glogower $E|\xi\rangle = \xi|\xi\rangle$ when $|\xi| \longrightarrow 1$). In this finite-dimensional space it can be given a basis selecting an orthonormal subset of the states (4.8). Once a reference phase state, say $|\theta_0\rangle$, has been chosen this basis is given by

$$|\theta_m\rangle = |\theta_0 + 2\pi m/\kappa\rangle \qquad m = 0, \ldots, \kappa - 1. \tag{4.9}$$

Note that they can be obtained simply by the action of the ambiguity group (2.4) as

$$|\theta_m\rangle = V(m)|\theta_0\rangle. \tag{4.10}$$

With this basis a Hermitian phase operator is constructed

$$\phi_\theta = \sum_{m=0}^{\kappa-1} \theta_m |\theta_m\rangle\langle\theta_m| \tag{4.11}$$

which verifies some relations desirable for a description of phase.

We can again take, in much the same way we did in the previous example, the space of definition of these states and operators to be $\mathcal{V}$. In other words it is just to change $|n\rangle$ by $|e_n\rangle$ in (4.8). Using the same notation for simplicity, the phase operator Φ_θ acting on the whole space $\overline{\mathcal{H}} \otimes \mathcal{V}$ that results from this definition is

$$\Phi_\theta = I \otimes \phi_\theta. \tag{4.12}$$

With this relation we can add to $\bar{a}^\dagger \bar{a}$ an operator with physical meaning to give a complete set of commuting observables. The associated basis in $\mathcal{H}$ is

$$|n, \theta_m\rangle = U^\dagger |n\rangle |\theta_m\rangle \tag{4.13}$$

with $|\theta_m\rangle$ defined previously but keeping in mind that the space where are defined is $\mathcal{V}$.

The original definition of the phase states (4.8) correspond to the states $|0, \theta_m\rangle$. In the same way we can take the limit $\kappa \longrightarrow \infty$. However the $|n, \theta_m\rangle$ basis can be used to describe states with properties between the number and the phase states.

5. Conclusions

The proposal of enlarging the Hilbert space in order to recover the unitarity of the transformation relating two Hamiltonians of different spectra seems to introduce spurious degrees of freedom (ambiguity spin). We have shown that with a proper definition of variables in phase space that have a clear counterpart in quantum mechanics, this ambiguity spin can be interpreted in connection with the multiphoton operator formalism.

In this framework multiphoton squeezed states have a clearer interpretation and appear to give a way to generalize and define non-classical states of light. The work of Moshinsky and co-workers on the representations in quantum mechanics of arbitrary canonical transformations supplies the theoretical tool for the problem. If we remember the importance of the linear case, the relevance of the subject is clear and could become a powerful tool for many problems in quantum optics.

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